SIF2028 Formulas

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1.1 Harmonics

- $y = A_1 \sin(x)$ is the first/fundemental harmonic
- $y = A_n \sin(nx)$is the n^{th} harmonic

1.2 Periodic Function

A function is periodic if:

$$f(x+P) = f(x), \quad \text{where } P = \text{period} \tag{1.1}$$

1.3 Fourier Series

1.3.1 Arbitrary Period, $2L$: $(-L \leq x \leq L)$

Fourier series, $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\} \tag{1.2}$$

Constant a_0:

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx \tag{1.3}$$

Constant a_n:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \tag{1.4}$$

Constant b_n:

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \tag{1.5}$$

1.3.2 Period, 2π: $(-\pi \leq x \leq \pi)$

Fourier series, $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{ a_n \cos(nx) + b_n \sin(nx) \} \tag{1.6}$$

Constant a_0:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx \tag{1.7}$$

Constant a_n:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \tag{1.8}$$

Constant b_n:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \tag{1.9}$$

1.3.3 Period, T: $\left(-\frac{T}{2} \leq x \leq \frac{T}{2}\right)$

Angular velocity, ω:

$$\omega = \frac{2\pi}{T} \quad \text{and} \quad T = \frac{2\pi}{\omega} \tag{1.10}$$

Fourier series, $f(t)$:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(n\omega t) + b_n \sin(n\omega t)\} \tag{1.11}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right)\right\} \tag{1.12}$$

Constant a_0:

$$a_0 = \frac{4}{T}\int_0^{T/2} f(t)\,dt \quad \text{or} \quad = \frac{2\omega}{\pi}\int_0^{2\pi/\omega} f(t)\,dt \tag{1.13}$$

Constant a_n:

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega t)\,dt \tag{1.14}$$

$$= \frac{2\omega}{\pi}\int_0^{2\pi/\omega} f(t)\cos(n\omega t)\,dt$$

Constant b_n:

$$b_n = \frac{4}{T}\int_0^{T} f(t)\sin(n\omega t)\,dt \tag{1.15}$$

$$= \frac{2\omega}{\pi}\int_0^{2\pi/\omega} f(t)\sin(n\omega t)\,dt$$

1.4 Dirichlet Conditions

Fourier series expansion of $f(x)$ converges to:

$$\boxed{\begin{array}{ll} a) & f(\alpha),\ \text{if } x = a \text{ is a point of continuity} \\ b) & \frac{1}{2}\left[\lim\limits_{x\to\alpha^-} f(x) + \lim\limits_{x\to\alpha^+} f(x)\right],\ \text{if } x = \alpha \text{ is a point of finite discontinuity.} \end{array}}$$

- If Dirichlet's conditions are satisfied, convergence of Fourier series to $f(x)$ is guaranteed.
- However, it is *sufficient but not necessary* for convergence.
- Fourier series converges to the mid-point of jump:

$$\boxed{\lim_{x\to\alpha^-} f(x) = \ell_1, \quad \lim_{x\to\alpha^+} f(x) = \ell_2, \quad \frac{1}{2}(\ell_1 + \ell_2)} \tag{1.16}$$

1.5 Odd Even Functions

Even function: $f(-x) = f(x)$; symmetrical about y-axis.
Odd function: $f(-x) = -f(x)$; symmetrical about origin.

1.5.1 Product of Odd and Even Function

$$(\text{even}) \times (\text{even}) = (\text{even}), \quad (\text{odd}) \times (\text{odd}) = (\text{even}), \quad (\text{odd}) \times (\text{even}) = (\text{odd}) \tag{1.17}$$

1.5.2 Sine Series and Cosine Series

We can simplify calculation of Fourier Series by considering whether it is even or odd:

1. If $f(x) =$ even, the series contains *cosine terms only*:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx, \quad b_n = 0 \tag{1.18}$$

2. If $f(x) =$ odd, the series contains *sine terms only*:

$$a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \tag{1.19}$$

1.6 Half-Range Series

For a function with period $= 2\pi$ and defined only in the range of $0 < T < \pi$. We can consider it to be half of an *even* function or *odd* function.

1.6.1 Period, T

1. Even function: Half-Range Cosine Series

$$a_0 = \frac{4}{T}\int_0^{T/2} f(t)\,dt, \quad a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega t)\,dt, \quad b_n = 0 \tag{1.20}$$

2. Odd function: Half-Range Sine Series

$$a_0 = 0, \quad a_n = 0, \quad b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega t)\,dt \tag{1.21}$$

1.6.2 Series Containing only Odd or Even Harmonics

1. If $f(x) = f(x+\pi)$, then Fourier Series only contains *even harmonics.*

$$f(x) = f(x+\pi) = \frac{a_0}{2} + \{a_2\cos(2x) + a_4\cos(4x) + ...\} + \{b_2\sin(2x) + b_4\sin(4x) + ...\} \tag{1.22}$$

2. If $f(x) = -f(x+\pi)$, then Fourier Series only contains *odd harmonics.*

$$f(x) = -f(x+\pi) = \{a_1\cos(x) + a_3\cos(3x) + ...\} + \{b_1\sin(x) + b_3\sin(3x) + ...\} \tag{1.23}$$

1.7 Complex Fourier Series

Euler's Formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{1.24}$$

Common identities:

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{1.25}$$

1.7.1 Sinc Function

Sinc function:

$$\text{sinc}(x) = \frac{\sin x}{x} \tag{1.26}$$

L'Hopital of $\text{sinc}(x)$:

$$\lim_{x\to 0} \frac{d}{dx}\left[\frac{\sin(x)}{x}\right] = \lim_{x\to 0} \frac{\cos(x)}{1} = 1 \tag{1.27}$$

1.7.2 Period, T

Complex Fourier Series, $f(t)$:

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ \left(\frac{a_n - ib_n}{2}\right) e^{in\omega_0 t} \cdot \left(\frac{a_n + ib_n}{2}\right) e^{-in\omega_0 t} \right\} \\ &= c_0 + \sum_{n=1}^{\infty} \left\{ c_n e^{in\omega_0 t} \cdot c_n^* e^{-in\omega_0 t} \right\} \end{aligned}$$

Simplified $f(t)$:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t} \tag{1.28}$$

Coefficient c_n (where $c_n \in \mathbb{C}$):

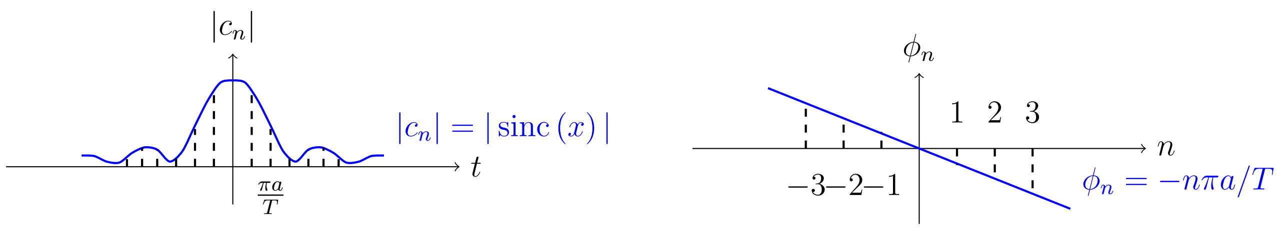
$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, e^{-in\omega_o t} dt \tag{1.29}$$

1.7.3 Complex Spectra

In general, c_n can be written as:

$$c_n = |c_n|\, e^{i\phi_n} \tag{1.30}$$

- These complex coefficients constitute a ***discrete complex spectrum.***
- c_n represents the ***spectral coefficient*** of the n^{th} harmonic.
- $|c_n|$ represents an ***amplitude spectrum*** which tells us the *magnitude each the harmonic has.*
- ϕ_n is the ***phase spectrum*** which tells us the *phase of each harmonic relative* to the *fundamental harmonic frequency* ω_0.



(a) Amplitude Spectrum, $|c_n|$

(b) Phase Spectrum, ϕ_n

Figure 1.1: Amplitude and Phase Spectrum [different shapes for each function $f(x)$]

1.7.4 The Two Domains

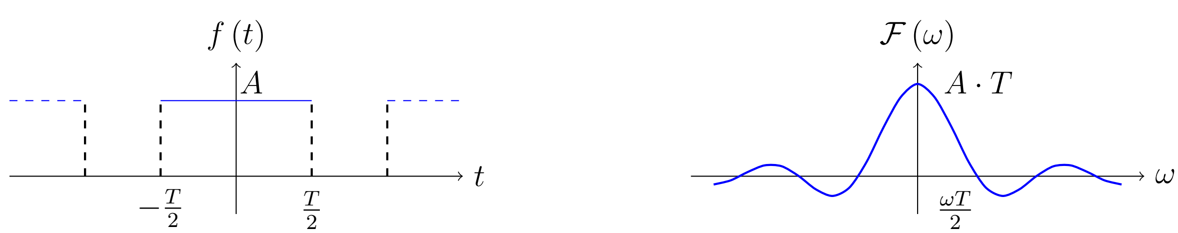
- **Waveform** is described in terms of behaviour in *time*, t.
- **Spectrum** is described in terms of behaviour *relative to frequency*, $\omega = 2\pi f$.
- Since t and ω form two domains of definition of our function, any information from one domain can be equally obtained within the other.
- For example, **power content** of periodic function $f(t)$ of period T defined in the *time domain* and *frequency domain* respectively:

$$\frac{1}{T}\int_{-T/2}^{T/2} (f(t))^2 \, dt \quad \text{and} \quad \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{1.31}$$

1.7.5 Continuous Spectra

- In Fourier series, distance between neighbouring harmonics in complex spectra is the fundamental frequency $\omega_0 = \frac{2\pi}{T}$.
- As $T \to \infty$, so $\omega_0 \to 0$. Means as the period increase, the space between lines in the spectrum decrease and eventually merge into a continuous spectrum. So, for large T:

$$n\omega_0 \approx n\delta\omega, \quad \text{and as } T \to \infty \Rightarrow n\delta\omega \to \omega \tag{1.32}$$
$$\text{where } \omega = \text{continuous frequency variable}$$



(a) Discrete time function, $f(t)$

(b) Continuous amplitude spectrum, $\mathcal{F}(\omega)$

Figure 1.2: Discrete and continuous spectrum [different shapes for each function $f(t)$]

Using this result, we can derive the Fourier Transform equation.

1.8 Fourier Transform

1.8.1 Fourier's Integral Theorem

Given function $f(t)$ with derivatives $f'(t)$ where

1. $f(t)$ and $f'(t)$ are piecewise continuous in every finite interval.
2. $f(t)$ is absolutely integrable in $(\infty, -\infty)$, that is $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$ (finite).

Fourier Transform, $\mathcal{F}(\omega)$:

$$\mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-iwt} dt \tag{1.33}$$

Inverse Fourier Transform, $f(\omega)$:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(\omega)\, e^{i\omega t} d\omega \tag{1.34}$$

1.8.2 Properties of Fourier Transform

1. Linearity

$$\mathscr{F}\left[\alpha_1 f_1(t) + \alpha_2 f_2(t)\right] = \alpha_1 \mathscr{F}\left[f_1(t)\right] + \alpha_2 \mathscr{F}\left[f_2(t)\right] \tag{1.35}$$
$$\text{where } \alpha_1, \alpha_2 = \text{any constants}$$

2. Time Shifting

If $\mathscr{F}[f(t)] = \mathcal{F}(\omega)$, then:

$$\mathscr{F}\left[f(t - t_0)\right] = e^{i\omega t_0} \mathcal{F}(w) \tag{1.36}$$

3. Frequency Shifting

If $\mathscr{F}[f(t)] = \mathcal{F}(\omega)$, then:

$$\mathscr{F}\left[f(t)\, e^{i\omega_0 t}\right] = \mathcal{F}(\omega - \omega_0) \tag{1.37}$$

4. Time Scaling

If $\mathscr{F}[f(t)] = \mathcal{F}(\omega)$, then:

$$\mathscr{F}\left[f(kt)\right] = \frac{1}{|k|} \mathcal{F}\left(\frac{\omega}{k}\right) \tag{1.38}$$

5. Symmetry

If $\mathscr{F}[f(t)] = \mathcal{F}(\omega)$, then:

$$\mathscr{F}\left[\mathcal{F}(t)\right] = f(-\omega) \tag{1.39}$$

6. Differentiation

If $f(t) \to 0$ as $t \to \pm\infty$ and if $\mathscr{F}[f(t)] = \mathcal{F}(\omega)$, then:

$$\mathscr{F}[f'(t)] = i\omega\mathcal{F}(\omega)$$

More generally:

$$\mathscr{F}\left[f^{(n)}(t)\right] = (i\omega)^n \mathcal{F}(\omega) \tag{1.40}$$

1.8.3 Alternative Forms of Fourier Transform

Alternate form 1:

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} \mathcal{F}(\omega)\, e^{i\omega t} d\omega \\ \mathcal{F}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt \end{aligned} \tag{1.41}$$

Alternate form 2:

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(\omega)\, e^{i\omega t} d\omega \\ \mathcal{F}(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt \end{aligned} \tag{1.42}$$

Alternate form 3:

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} \mathcal{F}(\omega)\, e^{i2\pi\omega t} d\omega \\ \mathcal{F}(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\omega t} dt \end{aligned} \tag{1.43}$$

1.9 Special Transforms

1.9.1 Odd, Even, Sine and Cosine

If $f(t)$ is odd or even, can use $\mathcal{F}_s$ and $\mathcal{F}_c$ respectively. Notice that values of *odd* & $\mathcal{F}_s$ and *even* & $\mathcal{F}_c$ are $\mathbb{C}$ and $\mathbb{R}$ respectively.

Odd functions:

$$\mathcal{F}(\omega) = -i\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin(\omega t)\, dt \tag{1.44}$$

Even functions:

$$\mathcal{F}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos(\omega t)\, dt \tag{1.45}$$

Fourier Sine Transform $\mathcal{F}_s$:

$$\mathcal{F}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin(\omega t)\, dt \tag{1.46}$$

Fourier Cosine Transform $\mathcal{F}_c$:

$$\mathcal{F}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos(\omega t)\, dt \tag{1.47}$$

Fourier Transform when $f(t) = e^{-qt}$:

$$\mathcal{F}_s = \sqrt{\frac{2}{\pi}} \cdot \frac{\omega}{\omega^2 + q^2} \quad \text{and} \quad \mathcal{F}_c = \sqrt{\frac{2}{\pi}} \cdot \frac{q}{\omega^2 + q^2}$$

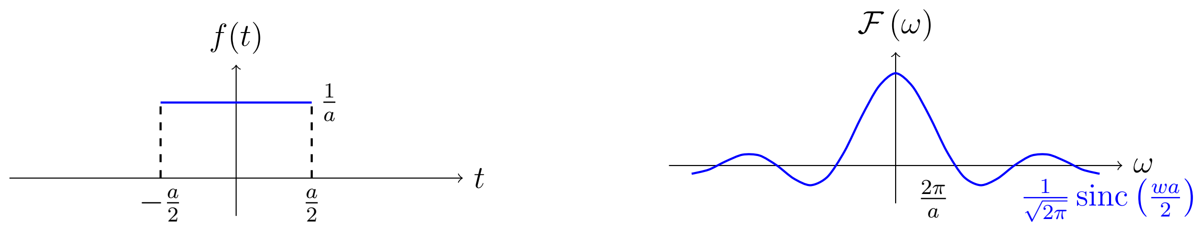
1.9.2 Top-Hat Function

Denoted by $\Pi_a(t)$ and is defined by:

$$f(x) = \begin{cases} 0, & t < -\frac{a}{2} \\ \frac{1}{a}, & -\frac{a}{2} < t < \frac{a}{2} \\ 0, & \frac{a}{2} < t \end{cases} \tag{1.48}$$

Fourier Transform $\mathcal{F}(\omega)$:

$$\mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}} \operatorname{sinc}\left(\frac{\omega a}{2}\right) \tag{1.49}$$



(a) Top-hat function, $\Pi_a(t)$

(b) Fourier transform of $\Pi_a(t) = \operatorname{sinc}\left(\frac{wa}{2}\right)$

Figure 1.3: Top-hat function, Π_a and its fourier transform, $F(\omega)$

It is useful because it can be used to select any segment of any function. For example, to select segment $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ of function $\cos(t)$:

$$\Pi_\pi(t-\pi) = \begin{cases} 0, & t - \pi < -\frac{\pi}{2} \\ \frac{1}{\pi}, & -\frac{\pi}{2} < t - \pi < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < t - \pi \end{cases}$$

$$\Pi_\pi(t-\pi) = \begin{cases} 0, & t < \frac{\pi}{2} \\ \frac{1}{\pi}, & \frac{\pi}{2} < t < \frac{3\pi}{2} \\ 0, & \frac{3\pi}{2} < t \end{cases}$$

$$\pi\Pi_\pi(t-\pi)\cos(t) = \begin{cases} \cos(t), & \frac{\pi}{2} < t < \frac{3\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

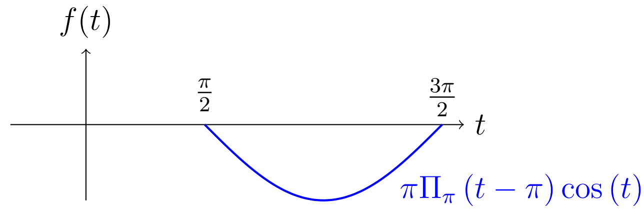


Figure 1.4: Top-hat function used to select segment of $\cos(t)$.

1.9.3 Dirac Delta Function

Is a ***unit area pulse***. Often used to represent force acting for a very brief period of time.

$$\lim_{a\to 0}\int_{-\infty}^{\infty}\{\Pi_a(t)\}dt = \int_{-\infty}^{\infty}\delta(t)\,dt = 1 \tag{1.50}$$

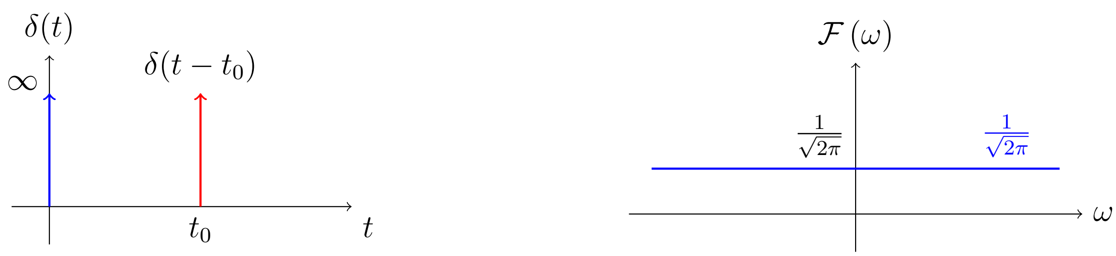
Therefore, we accept validity of integral:

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0) \tag{1.51}$$

Fourier Transform $\mathcal{F}(\omega)$:

$$\mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}} \tag{1.52}$$

Like the top-hat function, selects only *part of* $f(t)$ over which is non-zero, namely at $t = t_0$.



(a) Dirac delta function, $\delta(t)$

(b) Fourier transform of $\delta(t) = \frac{1}{\sqrt{2\pi}}$

Figure 1.5: Dirac delta function, $\delta(t)$ and its fourier transform, $F(\omega)$

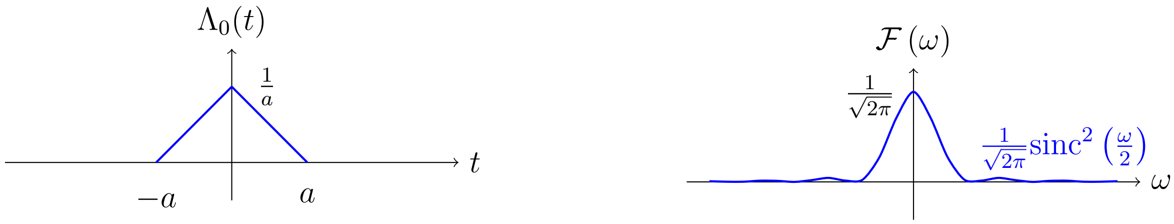
1.9.4 Triangle Function

Defined by the equation:

$$\Lambda_a(t) = \begin{cases} (a+t)/a^2, & -a < t < 0 \\ (a-t)/a^2, & 0 < t < a \\ 0, & |t| > a \end{cases} \tag{1.53}$$

Fourier Transform $\mathcal{F}(\omega)$:

$$\mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}}\mathrm{sinc}^2\left(\frac{\omega}{2}\right) \tag{1.54}$$



(a) Triangle function $\Lambda_a(t)$. **(b)** Fourier transform of $\Lambda_a(t) = \mathrm{sinc}^2\left(\frac{w}{2}\right)$

Figure 1.6: Triangle function, $\Lambda_a(t)$ and its fourier transform, $F(\omega)$

1.9.5 Heaviside Unit Step Function

Is defined as $u(t)$ where:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \tag{1.55}$$

Fourier Transform $\mathcal{F}(\omega)$:

$$\begin{aligned} \mathcal{F}(\omega) &= \frac{1}{\sqrt{2\pi} i\omega} - \left\{1 - \lim_{t\to\infty}\left[e^{-i\omega t}\right]\right\} \\ &= \frac{1}{\sqrt{2\pi}}\left\{\pi\delta(\omega) + \frac{1}{j\omega}\right\} \end{aligned} \tag{1.56}$$

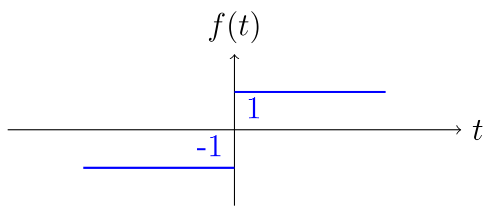


Figure 1.7: Heaviside unit step function $u(t)$

1.10 Convolution

Convolution of two functions $f(t)$ and $g(t)$ defined as:

$$\boxed{f(t) * g(t) = \int_{-\infty}^{\infty} f(x)\, g(t-x)\, dx = h(t)} \tag{1.57}$$

where $*$ denotes convolution

1.10.1 Properties of Convolution

1. Commutativity: $f * g = g * f$
2. Associativity: $f * (g * h) = (f * g) * h$
3. Distributivity: $f * (g + h) = (f * g) + (f * h)$

1.10.2 Convolution Theorem

If $\mathcal{F}(\omega)$ and $\mathcal{G}(\omega)$ are Fourier transforms of $f(t)$ and $g(t)$ respectively, then:

1. Fourier transform of the **convolution of f (t)** and **g (t)** is *equal* to the *product of the individual Fourier transforms*:

$$\mathscr{F}[f(t) * g(t)] = \sqrt{2\pi}\mathcal{F}(\omega)\mathcal{G}(\omega) \tag{1.58}$$

$$\mathscr{F}^{-1}[\mathcal{F}(\omega)\mathcal{G}(\omega)] = \frac{1}{\sqrt{2\pi}}[f(t) * g(t)] \tag{1.59}$$

2. Fourier transform of the **product of f (t) g (t)** is *equal* to the *convolution of the individual Fourier transforms*:

$$\mathscr{F}[f(t)g(t)] = \frac{1}{\sqrt{2\pi}}\mathcal{F}(\omega) * \mathcal{G}(\omega) \tag{1.60}$$

$$\mathscr{F}^{-1}[\mathcal{F}(\omega) * \mathcal{G}(\omega)] = \sqrt{2\pi}f(t)g(t) \tag{1.61}$$

They provide useful methods to find inverse transforms, especially in *Laplace Transforms.*

Table 1.1: Table of Transformations

Num.	Fourier, $f(t)$	Fourier Transform, $\mathcal{F}(\omega)$
1.	$f(t) = \begin{cases} 1 & \text{if } -a/2 < t < a/2 \\ 0 & \text{otherwise} \end{cases}$	$\mathcal{F}(\omega) = \frac{a}{\sqrt{2\pi}}\operatorname{sinc}(\omega a/2)$
2.	$f(t) = \begin{cases} 1 & \text{if } 0 < t < a \\ 0 & \text{otherwise} \end{cases}$	$\mathcal{F}(\omega) = \frac{ae^{-i\omega a/2}}{\sqrt{2\pi}}\operatorname{sinc}(\omega a/2)$
3.	$\Pi_a(t) = \begin{cases} 1/a & \text{if } -a/2 < t < a/2 \\ 0 & \text{otherwise} \end{cases}$	$\mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}}\operatorname{sinc}(\omega a/2)$
4.	$f(t) = u(t)$	$\mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}}\left\{\pi\delta(\omega) + \frac{1}{iw}\right\}$
5.	$f(t) = e^{-at}\cdot u(t)$	$\mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}}\left\{\pi\delta(\omega + a) + \frac{1}{iw}\right\}$
6.	$f(t) = te^{-at}\cdot u(t)$	$\mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}(a + iw)^2}$
7.	$f(t) = \delta(t)$	$\mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}}$

1.11 Miscellaneous

Table 1.2: Table of Cosine and Sine Integrals

Num.	Cosine and Sine Integrals (Period, L)
1.	$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx = 0$
2.	$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) dx = 0$
3.	$\int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right) dx = L$
4.	$\int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right) dx = L$
5.	$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$
6.	$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$
7.	$\int_{-L}^{L} \cos \frac{m\pi x}{L} \sin\left(\frac{n\pi x}{L}\right) dx = 0$

2 Laplace Transforms

The Laplace transform (one-to-one function) is an integral that transforms real variable function $f(t)$ with a function $F(s)$ as follows:

Let $f(t)$ be a function defined over $[0, \infty)$. Then:

$$\mathcal{L}\{f(t)\} = \int_{t=0}^{\infty} f(t)\, e^{-st} dt = F(s) \tag{2.1}$$

where s is assumed to be $+ve$ and large to ensure interval converges. Also, $\mathcal{L}$ may be intepreted as an operator.

2.1 Existence of Laplace Transform

2.1.1 Exponential Order

A function f is said to be of *exponential* if exists a constant $K > 0$ and $a \neq 0$ such that:

$$|f(t)| \leq K e^{at}, \quad \text{for all } t \geq t_0 \tag{2.2}$$

2.1.2 Existence Theorem

A Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ defined by (2.1), exists for $s > a$ if:

1. $f(t)$ is piecewise continuous on interval $0 \leq t \leq t_0$ for any positive t_0.
2. $f(t)$ is of exponential order.

Table 2.1: Elementary Laplace Transforms

Num.	$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$	Condition on s
1.	a	$\frac{a}{s}$	$s > 0$
2.	$t^n, \quad n = 0, 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$	$s > 0$
3.	e^{at}	$\frac{1}{s-a}$	$s > a$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
6.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	$s > \|a\|$
7.	$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > \|a\|$

2.2 Properties of Laplace Transforms

2.2.1 Linearity

Laplace transform is a *linear transformation* which means it satisfy the following properties:

If $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$ exists, and if α and β are constants, then:

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\} \tag{2.3}$$

Notice that the transforms **cannot be multiplied together.** For that, we will need to use *convolution* of two expressions.

2.2.2 First Shift Theorem

Used to find Laplace transforms of functions multiplied by an exponential factor.

If $\mathcal{L}\{f(t)\} = F(s)$ and a is a constant, then:

$$\mathcal{L}\left\{e^{at} \cdot f(t)\right\} = F(s-a) \tag{2.4}$$

2.2.3 Differentiation of a Transform

Relates operations in t domain to those in transformed s domain. It is known as *differentiation of a transform* or sometimes known as *multiplication by t property.*

If $\mathcal{L}\{f(t)\} = F(s)$, then for $n = 1, 2, 3, ...$

$$\mathcal{L}\{t^n \cdot f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)] \tag{2.5}$$

2.2.4 Integration of a Transform

It is known as *integration of a transform* or sometimes known as *division by t property.*

If $\mathcal{L}\{f(t)\} = F(s)$ and $\lim_{t\to 0} \frac{f(t)}{t}$ exists, then:

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds \tag{2.6}$$

2.2.5 Laplace Transform of an Integral

If $\mathcal{L}\{f(t)\} = F(s)$, then:

$$\mathcal{L}\left\{\int_0^t f(u)\, du\right\} = \frac{F(s)}{s} \tag{2.7}$$

Table 2.2: First Shift Theorem, Differentiation & Integration of Laplace Transforms and Laplace Transform of an Integral

Num.	$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1.	$e^{at} \cdot f(t)$	$F(s-a)$
2.	$t^n \cdot f(t)$	$(-1)^n \dfrac{d^n}{ds^n}[F(s)]$
3.	$\dfrac{f(t)}{t}$	$\displaystyle\int_s^\infty F(s)\,ds$
4.	$\displaystyle\int_0^t f(u)\,du$	$\dfrac{F(s)}{s}$

2.3 Inverse Laplace Transform

If $\mathcal{L}\{f(t)\} = F(s)$, then $f(t)$ is called inverse Laplace transform of $F(s)$ and is written as:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \tag{2.8}$$

The operator $\mathcal{L}^{-1}$ is known as the operator of inverse Laplace transform. Also, $\mathcal{L}^{-1} \neq \frac{1}{\mathcal{L}}$.

Table 2.3: Inverse Laplace Transforms

Num.	$F(s)$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$
1.	$\dfrac{a}{s}$	a
2.	$\dfrac{n!}{s^{n+1}}$	$t^n, \quad n = 0, 1, 2, ...$
3.	$\dfrac{1}{s-a}$	e^{at}
4.	$\dfrac{a}{s^2+a^2}$	$\sin(at)$
5.	$\dfrac{s}{s^2+a^2}$	$\cos(at)$
6.	$\dfrac{a}{s^2-a^2}$	$\sinh(at)$
7.	$\dfrac{s}{s^2-a^2}$	$\cosh(at)$

2.4 Properties of Inverse Laplace Transforms

2.4.1 Linearity

This property is valid for n terms.

If $\mathcal{L}^{-1}\left\{F\left(s\right)\right\}=f\left(t\right)$ and $\mathcal{L}^{-1}\left\{\mathcal{G}\left(s\right)\right\}=g\left(t\right)$, and if α and β are constants then:

$$\mathcal{L}^{-1}\left\{\alpha F\left(s\right)+\beta\mathcal{G}\left(s\right)\right\}=\alpha\mathcal{L}^{-1}\left\{F\left(s\right)\right\}+\beta\mathcal{L}^{-1}\left\{\mathcal{G}\left(s\right)\right\}=\alpha f\left(t\right)+\beta g\left(t\right) \tag{2.9}$$

2.4.2 First Shift Property

If $\mathcal{L}^{-1}\left\{F\left(s\right)\right\}=f\left(t\right)$ and a is a constant, then:

$$\mathcal{L}^{-1}\left\{F\left(s-a\right)\right\}=e^{at}\mathcal{L}^{-1}\left\{F\left(s\right)\right\}=e^{at}f\left(t\right) \tag{2.10}$$

2.4.3 Second Shift Property

If $\mathcal{L}^{-1}\left\{F\left(s\right)\right\}=f\left(t\right)$ and a is a constant, then:

$$\mathcal{L}^{-1}\left\{e^{-as}F\left(s\right)\right\}=f\left(t-a\right)H\left(t-a\right) \tag{2.11}$$

where $H\left(t\right)=$unit step function

2.4.4 Differentiation of a Transform

If $\mathcal{L}^{-1}\left\{F\left(s\right)\right\}=f\left(t\right)$, then:

$$\mathcal{L}^{-1}\left\{F\left(s\right)\right\}=-\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left[F\left(s\right)\right]\right\} \tag{2.12}$$

2.4.5 Integration of a Transform

If $\mathcal{L}^{-1}\left\{F\left(s\right)\right\}=f\left(t\right)$, then:

$$\mathcal{L}^{-1}\left\{F\left(s\right)\right\}=t\cdot\mathcal{L}^{-1}\left\{\int_{s}^{\infty}F\left(s\right)ds\right\} \tag{2.13}$$

2.5 Partial Fractions

Consider the expression of the form:

$$\frac{N(s)}{D(s)} \tag{2.14}$$

where $N(s)$ and $D(s)$ are polynomials in degree s and degree of $D(s) > N(s)$.

2.5.1 General Rules for Partial Fraction

1. The degree of $D(s)$ must be greater than the degree of $N(s)$. If not, long division.
2. For each linear factor $(s+a)$ in the denominator, assume there to be a partial fraction of the form $\dfrac{A}{s+a}$ where A is a constant.
3. For each repeated linear factor $(s+a)^n$ in the denominator, assume there to be n partial fractions of the form:

$$\frac{A_1}{s+a} + \frac{A_2}{(s+a)^2} + \frac{A_3}{(s+a)^3} + ... + \frac{A_n}{(s+a)^n}$$

4. For each irreducible quadratic factor (s^2+ps+q) in the denominator, assume there to be partial fraction of the form $\dfrac{Ps+Q}{s^2+ps+q}$ where P and Q are constants.
5. For each irreducible factor $(s^2+ps+q)^n$, assume there to be n partial fractions of the form:

$$\frac{P_1 s + Q_2}{s^2+ps+q} + \frac{P_2 s + Q_2}{(s^2+ps+q)^2} + ... + \frac{P_n s + Q_n}{(s^2+ps+q)^n}$$

2.5.2 Cover Up Rule

A shortcut method to determine the value of constants $A_1, A_2, A_3, ...$. Can only be used if the *denominator* is a *product of linear factors*.

$$F(s) = \frac{s+17}{(s-1)(s+2)(s-3)} \equiv \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s-3}$$

$$A = \lim_{s\to 1} \frac{s+17}{(s+2)(s-3)} = -3, \quad B = \lim_{s\to -2} \frac{s+17}{(s-1)(s-3)} = 1$$

$$C = \lim_{s\to 3} \frac{s+17}{(s-1)(s+2)} = 2$$

For situations like the following, cover up rule can still be useful:

$$\frac{2x+1}{(x-1)^2(x-2)^2} = \frac{A}{x-1} + \frac{3}{(x-1)^2} + \frac{B}{x-2} + \frac{5}{(x-2)^2} \tag{2.15}$$

2.5.3 Useful Techniques

Situation 1

For situations of the form $\frac{s}{(s-a)^2+q}$, manipulate algebraically then apply *first shift theorem*:

$$\frac{s}{(s-a)^2+3} = \frac{(s-a)+a}{(s-a)^2+3} = \frac{(s-a)}{(s-a)^2+3} + \frac{a}{(s-a)^2+3} \tag{2.16}$$

Situation 2

For situations with $\frac{1}{s^2}$ in denominator, detach then apply *cover up rule* on the inner bracket:

$$\frac{4}{s^2(s+2)(s+3)} = \frac{1}{s}\left\{\frac{4}{s(s+2)(s+3)}\right\} \tag{2.17}$$

Next, apply *cover up rule* on the inner bracket.

$$\frac{1}{s}\left\{\frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}\right\} = \frac{1}{s}\left\{\frac{4}{6}\cdot\frac{1}{s} - \frac{2}{s+2} + \frac{4}{3}\cdot\frac{1}{s+3}\right\}$$

Then, multiply $\frac{1}{s}$ back into the inner bracket. Further simplify any remaining fraction into partial fractions:

$$\frac{2}{3}\cdot\frac{1}{s^2} - \frac{2}{s(s+2)} + \frac{4}{3}\cdot\frac{1}{s(s+3)}$$

2.6 Convolution Theorem

For the following product, We wish to determine $\mathcal{L}^{-1}\{F(s)\mathcal{G}(s)\}$:

$$F(s)\mathcal{G}(s), \quad \text{with inverses:} \quad \mathcal{L}^{-1}\{F(s)\} = f(t) \quad \text{and} \quad \mathcal{L}^{-1}\{\mathcal{G}(s)\} = g(t) \tag{2.18}$$

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $\mathcal{L}^{-1}\{\mathcal{G}(s)\} = g(t)$:

$$\mathcal{L}^{-1}\{F(s)\mathcal{G}(s)\} = \int_0^t f(u)\,g(t-u)\,du \tag{2.19}$$

Given the expression:

$$\frac{1}{(s+1)(s-2)} = \frac{1}{s+1}\cdot\frac{1}{s-2}$$

We choose:

$$F(s) = \frac{1}{s+1} \quad \text{and} \quad \mathcal{G}(s) = \frac{1}{s-2}$$

From which:

$$f(t) = e^{-t} \quad \text{and} \quad g(t) = e^{2t}$$
$$f(u) = e^{-u} \quad \text{and} \quad g(t-u) = e^{2(t-u)}$$

Hence:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\} = \int_0^t e^{-u}\cdot e^{2(t-u)}du = \frac{1}{3}\left(e^{2t} - e^{-t}\right)$$

2.7 Solutions of DE by Laplace Transforms

Laplace transform is an effective method to solve ODE, especially nonhomogeneous equations with input function in the form of step or delta function.

2.7.1 Transforms of Derivatives

$$
\begin{aligned}
\text{if} \quad \mathcal{L}\left\{y(t)\right\} &= Y(s)\,, \text{ then} \\
\mathcal{L}\left\{y'(t)\right\} &= sY(s) - y(0) \\
\mathcal{L}\left\{y''(t)\right\} &= s^2Y(s) - sy(0) - y'(0) \\
\mathcal{L}\left\{y'''(t)\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
&\vdots \\
\mathcal{L}\left\{y^{(n)}(t)\right\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - y^{n-1}(0)\,. \qquad (2.20)
\end{aligned}
$$

2.7.2 Initial Value Problem

A) Linear DE of First Order

Given initial value problem of ODE first order:

$$ay' + by = f(t)\,, \quad y(0) = y_0, \tag{2.21}$$

where a, b and y_0 are constants

Taking its Laplace transform:

$$a\mathcal{L}\left\{y'\right\} + b\mathcal{L}\left\{y\right\} = \mathcal{L}\left\{f(t)\right\}. \tag{2.22}$$

Using the theorem previously:

$$
\begin{aligned}
a\left[sY(s) - y(0)\right] + bY(s) &= F(s) \\
(as + b)\,Y(s) - ay_0 &= F(s)
\end{aligned}
$$

Obtain an algebraic equation in s. Thus:

$$Y(s) = \frac{F(s) + ay_0}{as + b} \tag{2.23}$$

Finally, perform inverse Laplace transform to obtain the solution in t:

$$y(t) = \mathcal{L}^{-1}\left\{Y(s) = \frac{F(s) + ay_0}{as + b}\right\} \tag{2.24}$$

B) Linear DE of Second Order

Given initial value problem of ODE second order:

$$ay''+by+cy=f\left(t\right),\quad y\left(0\right)=y_0\quad\text{and}\quad y'\left(0\right)=y_1 \tag{2.25}$$

where a, b, c, y_0 and y_1 are constants

Taking its Laplace transform:

$$a\mathcal{L}\left\{y''\right\}+b\mathcal{L}\left\{y'\right\}+c\mathcal{L}\left\{y\right\}=\mathcal{L}\left\{f\left(t\right)\right\}. \tag{2.26}$$

Using the theorem previously:

$$\begin{aligned} a\left[s^2Y\left(s\right)-sy\left(0\right)-y'\left(0\right)\right]+b\left[sY\left(s\right)-y\left(0\right)\right]+cY\left(s\right)&=F\left(s\right)\\ \left(as^2+bs+c\right)Y\left(s\right)-\left(as+b\right)y_0-ay_1&=F\left(s\right)\end{aligned}$$

Obtain an algebraic equation in s. Thus:

$$Y\left(s\right)=\frac{F\left(s\right)+\left(as+b\right)y_0+ay_1}{as^2+bs+c} \tag{2.27}$$

Finally, perform inverse Laplace transform to obtain the solution in t:

$$y\left(t\right)=\mathcal{L}^{-1}\left\{\frac{F\left(s\right)+\left(as+b\right)y_0+ay_1}{as^2+bs+c}\right\} \tag{2.28}$$

Table 2.4: Summary of Linear DE of First and Second Order

Step	First Order	Second Order
1.	$ay'+by=f\left(t\right),\quad y\left(0\right)=y_0,$	$ay''+by+cy=f\left(t\right),\quad y\left(0\right)=y_0,\ y'\left(0\right)=y_1$
2.	$Y\left(s\right)=\frac{F\left(s\right)+ay_0}{as+b}$	$Y\left(s\right)=\frac{F\left(s\right)+\left(as+b\right)y_0+ay_1}{as^2+bs+c}$
3.	$y\left(t\right)=\mathcal{L}^{-1}\left\{\frac{F\left(s\right)+ay_0}{as+b}\right\}$	$y\left(t\right)=\mathcal{L}^{-1}\left\{\frac{F\left(s\right)+\left(as+b\right)y_0+ay_1}{as^2+bs+c}\right\}$
4.	Solution $y\left(t\right)$ is found	Solution $y\left(t\right)$ is found

2.7.3 Boundary Value Problem

Approach is the same as for **Initial Value Problem**. However, since we do not have initial values such as $y\left(0\right)=1$ or $y'\left(0\right)=2$, rather we have $y\left(2\right)=\pi/2$ or $y'\left(5\pi\right)=4.2$. Therefore we set:

$$y\left(0\right)=\alpha,\quad y'\left(0\right)=\beta,\quad \ldots \tag{2.29}$$

Then, solve the ODE as usual but using the constants defined previously:

$$y\left(t\right)=\left(\alpha+1\right)e^t-e^{2t}+te^{2t} \tag{2.30}$$

Finally, substitute the boundary values into equation to determine value of constants:

$$y\left(1\right)=2e=\left(\alpha+1\right)e\Rightarrow\alpha=1. \tag{2.31}$$

2.8 Systems of Differential Equations

Simultaneous ODE involve *more than one dependent variable* such as $x(t)$ and $y(t)$. Therefore, Laplace transform is needed for each variable. The procedure then, is to solve the simultaneous equation for transformed variables $X(s)$ and $Y(s)$. Finally, invert to recover each dependent variables $x(t)$ and $y(t)$.

Given the following simultaneous equations:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x \quad \text{and} \quad x(0) = 1,\ y(0) = 2$$

Let $\mathcal{L}\{x(t)\} = X(s)$ and $\mathcal{L}\{y(t)\} = Y(s)$. Taking Laplace transform of both sides:

$$\begin{aligned} sX(s) - x(0) &= Y(s) \\ sY(s) - y(0) &= -X(s) \end{aligned} \tag{2.32}$$

Apply initial conditions and rearranging:

$$sX(s) - Y(s) = 1 \tag{i}$$
$$sY(s) + X(0) = 2 \tag{ii}$$

Solve for $X(s)$ by $s \times (i) + (ii)$ and $Y(s)$ by $s \times (ii) - (i)$:

$$X(s) = \frac{s}{s^2+1} + \frac{2}{s^2+1} \quad \text{and} \quad Y(s) = \frac{2s}{s^2+1} - \frac{1}{s^2+1}$$

Taking inverse Laplace transform, we obtain:

$$\begin{aligned} x(t) &= \cos(t) + 2\sin(t) \\ y(t) &= 2\cos(t) - \sin(t) \end{aligned}$$

* Note: Alternatively, $y(t)$ may also be solved by substituting $x(t)$ into the first equation.

$$y(t) = \frac{dx}{dt} = \frac{d}{dt^2}\left(x(t) = \cos(t) + 2\sin(t)\right) = -\sin(t) + 2\cos(2t) \tag{2.33}$$

3 Special Functions

3.1 Integral Functions

3.1.1 Gamma Function, $\Gamma(x)$

Defined by the integral:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad \text{convergent for } x > 0 \tag{3.1}$$

Properties:

1. Recurrence relation:
   $\Gamma(x+1) = x\Gamma(x), \quad \text{where } x = 1, 2, 3, \ldots$
2. In general:
   $\Gamma(x+1) = x!\Gamma(1) = x!, \quad \text{where } \Gamma(1) = 1$
3. Reverse recurrence relation:
   $\Gamma(x) = \dfrac{\Gamma(x+1)}{x}$
4. Negative values of x:
   a. Integers: $\Gamma(0) = \infty; \quad \Gamma(-x) = \pm\infty$
   b. Fraction ($x = \frac{n}{2}$): $\Gamma(-x) = \dfrac{\Gamma(-x+1)}{-x}$
5. Duplication formula:
   $\Gamma\left(n + \frac{1}{2}\right) = \dfrac{\Gamma(2n)\sqrt{\pi}}{2^{2n-1}\Gamma(n)}$
6. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = 2\displaystyle\int_0^\infty e^{-u^2} du$

i. Stirling formula:
   $n! \approx \sqrt{2\pi n}\, n^n e^{-n}, \quad \text{where } n \text{ is large}$

ii. $\displaystyle\int_0^\infty x^n \cdot e^{-ax} dx = \frac{\Gamma(n+1)}{a^{n+1}} = \frac{n!}{a^{n+1}}$

iii. $\displaystyle\int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$

iv. $\displaystyle\int_0^\infty x^m \cdot e^{-ax^2} dx = \frac{\Gamma\left(\frac{m+1}{2}\right)}{2a^{(m+1)/2}}$

x	$\Gamma(x)$
0·25	3·6256
0·50	1·7725
0·75	1·2254
1·00	1·0000
1·25	0·9064
1·50	0·8862
1·75	0·9191
2·00	1·0000
2·25	1·1330
2·50	1·3293

x	$\Gamma(x)$
2·75	1·6084
3·00	2·0000
3·25	2·5493
3·50	3·3234
3·75	4·4230
4·00	6·0000
4·25	8·2851
4·50	11·6318
4·75	16·5862
5·00	24·0000

Figure 3.1: Table of values of $\Gamma(x)$

x	0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0
$\Gamma(x)$	∞	1·772	1·000	0·886	1·000	1·329	2·000	3·323	6·000

x	−0·5	−1·5	−2·5	−3·5
$\Gamma(x)$	−3·545	2·363	−0·945	0·270

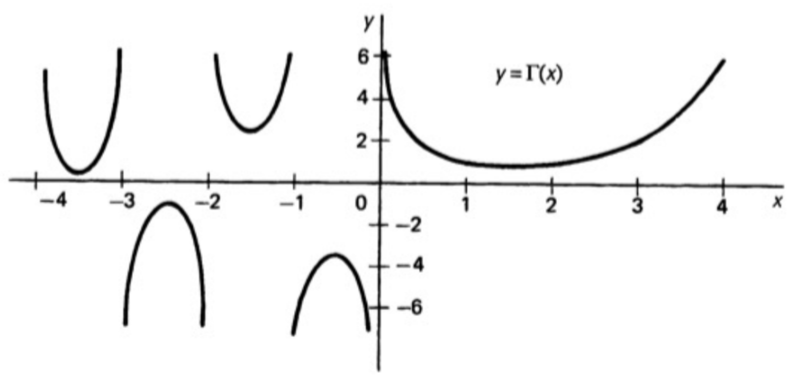


Figure 3.2: Values and plot of $y = \Gamma(x)$

3.1.2 Beta Function, $\mathcal{B}(m,n)$

Defined by the integral:

$$\mathcal{B}(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad \text{convergent for } m,n>0 \tag{3.2}$$

Properties:

1. Symmetry property:
   $\mathcal{B}(m,n) = \mathcal{B}(n,m)$
2. Trigonometric form:
   $\mathcal{B}(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}(\theta)\cos^{2m-1}(\theta)\,d\theta, \quad \left(\text{sub } x=\sin^2\theta\right)$
3. Reduction formula:
   $\mathcal{B}(m,n) = \dfrac{(m-1)(n-1)}{(m+n-1)(m+n-2)}\mathcal{B}(m-1,n-1)$
4. In general:
   $\mathcal{B}(m,n) = \dfrac{(m-1)!\,(n-1)!}{(m+n-1)!}$
5. $\mathcal{B}(k,1) = \mathcal{B}(1,k) = \dfrac{1}{k}$
6. $\mathcal{B}\left(\frac{1}{2},\frac{1}{2}\right) = \pi$

3.1.3 Relation between Gamma and Beta Functions

$$\mathcal{B}(m,n) = \mathcal{B}(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad m,n\in\mathbb{R} \tag{3.3}$$

Reduction Formula of Sines and Cosines

1. $\displaystyle\int_0^{\pi/2} \sin^n(x)\,dx = \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}(x)\,dx, \quad S_n = \frac{n-1}{n}S_{n-2}$
2. $\displaystyle\int_0^{\pi/2} \cos^n(x)\,dx = \frac{n-1}{n}\int_0^{\pi/2}\cos^{n-2}(x)\,dx, \quad C_n = \frac{n-1}{n}C_{n-2}$
3. $\displaystyle\int_0^{\pi/2} \sin^m(x)\cos^n(x) = \frac{m-1}{m+n}\int_0^{\pi/2}\sin^{m-2}(x)\cos^n(x)\,dx \equiv I_{m,n} = \frac{m-1}{m+n}I_{m-2,\ n}$
4. $\displaystyle\int_0^{\pi/2} \sin^m(x)\cos^n(x) = \frac{n-1}{m+n}\int_0^{\pi/2}\sin^{m}(x)\cos^{n-2}(x)\,dx \equiv I_{m,n} = \frac{n-1}{m+n}I_{m,\ n-2}$

3.2 Error Function

Defined as:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{3.4}$$

From the definition of $\Gamma\left(\frac{1}{2}\right)$, we can find the limits:

$$\lim_{x\to\infty} [\operatorname{erf}(x)] = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt = \frac{1}{\sqrt{2}} \cdot \Gamma\left(\tfrac{1}{2}\right) = 1 \tag{3.5}$$

Representing the exponential function in the integral with Maclaurin series:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(2n+1)} \tag{3.6}$$

Consequently, $\operatorname{erf}(x)$ is an odd function:

$$\operatorname{erf}(-x) = -\operatorname{erf}(x) \tag{3.7}$$

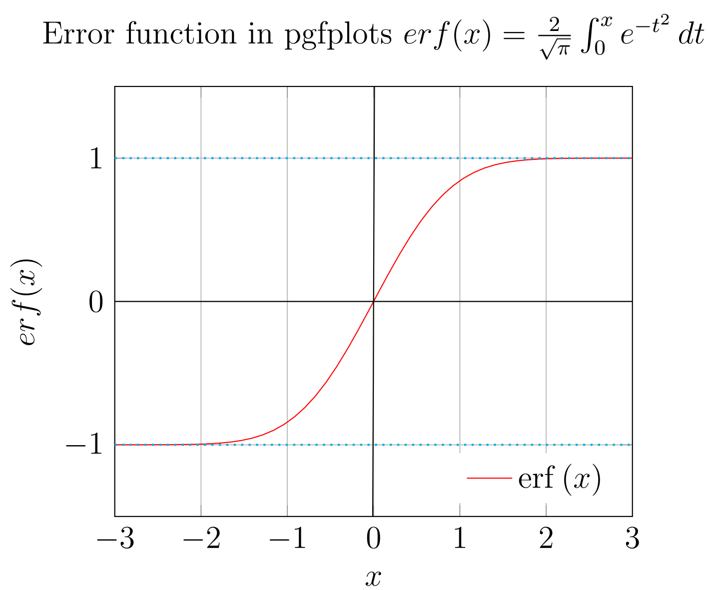


Figure 3.3: Plot of error function $\operatorname{erf}(x)$

3.2.1 Complementary Error Function, $\operatorname{erfc}(x)$

Defined as:

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \tag{3.8}$$

which is related to the Error function by relation:

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) \tag{3.9}$$

3.2.2 Relationship between Error and Gaussian Function

Gaussian integral is defined as:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt, \quad \text{where} \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} dt = 1 \tag{3.10}$$

For positive x. $\Phi(x)$ is related to $\operatorname{erf}(x)$ by:

$$\Phi(x) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) \tag{3.11}$$

3.3 Elliptic Functions

An *integral is elliptic* if the ***integrand is a rational function of*** x ***and*** $\sqrt{\mathbf{P}(\mathbf{x})}$.
(where $P(x)$ is a polynomial of degree 3 or 4)

$$\int_0^1 \frac{dx}{\sqrt{(1-2x^2)(4-3x^2)}} \tag{3.12}$$

3.3.1 Standard and Alternative Form of Elliptic Function

i) Of the First Kind

$$F(k,\phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}} \quad \text{or} \quad F(k,x) = \int_0^{x} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{3.13}$$

where $0 < k < 1$ and ($0 \le \phi \le \frac{\pi}{2}$ or $0 \le x \le 1$)

ii) Of the Second Kind

$$E(k,\phi) = \int_0^{\phi} \sqrt{1-k^2\sin^2(\theta)}\, d\theta \quad \text{or} \quad E(k,x) = \int_0^{x} \sqrt{\frac{1-k^2u^2}{1-u^2}} du \tag{3.14}$$

where $0 < k < 1$ and ($0 \le \phi \le \frac{\pi}{2}$ or $0 \le x \le 1$)

3.3.2 Complete Elliptic Functions

For Equations (3.13) and (3.14): *If* $\phi = \frac{\pi}{2}$, the ***integral is said to be complete.*** Then:

$$F\left(k, \tfrac{\pi}{2}\right) \text{ denoted by } F(k) \quad \text{and} \quad E\left(k, \tfrac{\pi}{2}\right) \text{ denoted by } E(k) \tag{3.15}$$

$$F(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}} \quad \text{and} \quad E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2(\theta)\, d\theta}$$

3.3.3 Useful Techniques

Situation 1

For integrals of the form:

$$I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1+4\sin^2\theta}} \tag{3.16}$$

Solve by letting $\theta = \frac{\pi}{2} - \psi \Rightarrow \sin\theta = \cos\psi$ and changing the limits. Then:

$$I = \int_{\frac{\pi}{2}}^{0} \frac{-d\psi}{\sqrt{1+4\cos^2\psi}} = \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1+4\left[1-\sin^2\psi\right]}} = \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{5-4\sin^2\psi}}$$

Situation 2

For integrals of the form:

$$I = \int_0^{2} \frac{dt}{\sqrt{(4-t^2)(9-t^2)}} \tag{3.17}$$

Select denominator $(4-t^2)$. Then let $t = 2\sin\theta \Rightarrow dt = 2\cos\theta\ d\theta$ and changing the limits:

$$I = \int_0^{\frac{\pi}{2}} \frac{2\cos\theta\ d\theta}{\sqrt{\left(4-4\sin^2\theta\right)\left(9-4\sin^2\theta\right)}} = \int_0^{\frac{\pi}{2}} \frac{2\cos\theta\ d\theta}{2\cos\theta\cdot\sqrt{9-4\sin^2\theta}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{9-4\sin^2\theta}}$$

4 Power Series and ODE

4.1 Higher Derivatives

4.1.1 Common Identities

Table 4.1: Highier Derivatives Common Identities

Num.	$y(x)$	$y^{(n)}(x)$
1.	x^a	$\dfrac{a!}{(a-n)!} \cdot x^{a-n}$
2.	e^{ax}	$a^n \cdot e^{ax}$
3.	$\ln x$	$(-1)^{n-1} \cdot \dfrac{(n-1)!}{x^n}$
4.	$\sin(x)$	$\sin\left(x + \dfrac{n\pi}{2}\right)$
5.	$\cos(x)$	$\cos\left(x + \dfrac{n\pi}{2}\right)$
6.	$\sin(ax)$	$a^n \cdot \sin\left(ax + \dfrac{n\pi}{2}\right)$
7.	$\cos(ax)$	$a^n \cdot \cos\left(ax + \dfrac{n\pi}{2}\right)$
8.	$\sinh(ax)$	$\dfrac{a^n}{2}\left\{[1+(-1)^n]\sinh(ax) + [1-(-1)^n]\cosh(ax)\right\}$
9.	$\cosh(ax)$	$\dfrac{a^n}{2}\left\{[1-(-1)^n]\sinh(ax) + [1+(-1)^n]\cosh(ax)\right\}$

4.1.2 Leibnitz Theorem - n^{th} Derivative of Product of Two Functions

Given that $y = uv$ where u and v are functions of x, then (Pascal's Triangle):

$$y^{(n)} = \sum_{r=0}^{n} {}^nC_r u^{(n-r)} v^{(r)}, \quad \text{where } {}^nC_r = \frac{n!}{r!\,(n-r)!} \tag{4.1}$$

Choices of Functions for u and v

For the product $y = uv$, the function taken as:

i. $u \sim n^{\text{th}}$ derivative can be easily obtained.

ii. $v \sim$ derivatives reduce to zero after a small number of differentiation.

4.2 Power Series Solution

Equations of the following form can **only be solved** by expressing it as an *infinite power series* of x. We need the following two methods to obtain the infinite series.

$$\boxed{\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0} \tag{4.2}$$

4.2.1 Leibnitz - Maclaurin Method

Leibnitz Theorem

Expanding Equation 4.1:

$$\begin{aligned} y^{(n)} &= u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + ... \\ &+ \frac{n(n-1)...(n-r+1)}{r!}u^{(n-r)}v^{(r)} + ... + uv^{(n)} \end{aligned} \tag{4.3}$$

Maclaurin's Series

For a valid solution, series obtained must converge (apply ratio test).

$$y = (y)_0 + x(y')_0 + \frac{x^2}{2!}(y'')_0 + ... + \frac{x^n}{n!}\left(y^{(n)}\right)_0 + ... \tag{4.4}$$

where $\left(y^{(n)}\right)_0$ is the value of n^{th}derivative of y when $x = 0$

Guideline Express ODE as Power Series

1. Differentiate given equation n times using Leibnitz theorem.
2. Rearrange result to obtain **recurrence relation** (at $x = 0$).
3. Determine values of derivatives at $x = 0$ (usually) in terms of $(y)_0$ and $(y')_0$.
4. Substitute in Maclaurin's expansion for $y = f(x)$.
5. Simplify results and apply boundary conditions.

4.2.2 Frobenius' Method

Sometimes power series do not converge. A more general method is to ***assume trial solution*** of the form:

$$y = x^c\left\{a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_rx^r + ...\right\}, \quad \text{where } a_0 \neq 0 \tag{4.5}$$

This type of equation can be **solved by** method of form:

$$y'' + P(x)y' + Q(x)y = 0 \tag{4.6}$$

where $P(x)$ and $Q(x)$ are functions of x.

Conditions to be Satisfied

1. If functions P and Q are both finite when $x = 0$, it is called ***ordinary point***.
$$P(0),\ Q(0) \neq \infty$$
2. If xP and x^2Q remain finite at $x = 0$, then $x = 0$ is called ***regular singular point***.
$$P \neq \frac{1}{x^2}, \frac{1}{x^3}, \ \ldots \quad \text{and} \quad Q \neq \frac{1}{x^3}, \frac{1}{x^4}, \ \ldots$$
3. However, if P and Q do not satisfy either conditions, then $x = 0$ is called ***irregular singular point*** of the equation and Frobenius' cannot be applied.

Solution of DE by Frobenius' Method

To solve given equation, need to find ***coefficients*** $a_0, a_1, \ \ldots$ and index c in the *trial solution.*

1. Differentiate the *trial series* (Equation 4.5) as required.
$$\begin{aligned} y &= a_0x^c + a_1x^{c+1} + a_2x^{c+2} + \ldots + a_rx^{c+r} + \ldots \\ y' &= a_0cx^{c-1} + a_1(c+1)x^c + a_2(c+2)x^{c+1} + \ldots + a_r(c+r)x^{c+r-1} + \ldots \\ y'' &= a_0c(c-1)x^{c-2} + a_1c(c+1)x^{c-1} + a_2(c+1)(c+2)x^c + \ldots \\ &\quad + a_r(c+r-1)(c+r)x^{c+r-2} + \ldots \end{aligned} \tag{4.7}$$
2. Substitute the results into the given DE. For example, $2xy'' + y' + y{=}0$:
$$\begin{aligned} y &= a_0x^c + a_1x^{c+1} + a_2x^{c+2} + \ldots + a_rx^{c+r} + \ldots \\ y' &= a_0cx^{c-1} + a_1(c+1)x^c + a_2(c+2)x^{c+1} + \ldots + a_r(c+r)x^{c+r-1} + \ldots \\ 2xy'' &= a_0c(c-1)x^{c-1} + 2a_1c(c+1)x^c + 2a_2(c+1)(c+2)x^{c+1} + \ldots \\ &\quad + 2a_r(c+r-1)(c+r)x^{c+r-1} + \ldots \end{aligned} \tag{4.8}$$
3. Equate coefficients of corresponding powers of x on each side of the equation by adding all three equations above. For example, given equation $2xy'' + y' + y = 0$:
$$\text{Coefficient } \left[x^{c-1}\right] : 2a_0c(c-1) + a_0c = a_0c(2c-1) = 0 \ (\mathcal{RHS}) \tag{4.9}$$
$$\text{Coefficient (general) } \left[x^{c+r}\right] : a_{r+1}\{(c+r+1)(2c+2r+1)\} + a_r = 0 \tag{4.10}$$

4.2.3 Indical Equation

Equation formed from the **coefficient of lowest powers of x**, from which the *values of c can be obtained.* From previous example:

$$a_0c(2c-1) = 0 \Longrightarrow c = 0 \text{ or } c = \frac{1}{2} \tag{4.11}$$

Additional Conditions

1. If c_1 and c_2 differ by a quantity **NOT** an *integer*, then two independent solutions, $y = u(x)$ and $y = v(x)$ are obtained. Then, general solution is $y = Au + Bv$.
2. If c_1 and c_2 differ by an **integer**: $c_2 = c_1 + n$, (where $n \in \mathbb{Z}$) and if one coefficient (a_r) is determined when $c = c_1$. Then the *complete solution* is given by using values of c (because, using $c = c_1 + n$ gives a series which is a simple multiple of one of the series in the first solution).
3. If roots $c = c_1$ and $c = c_1 + n$ of the indicial equation **differ by an integer** and **one coefficient** (a_r) **becomes infinite** when $c = c_1$, the series is rewritten with a_0 replaced by $k(c - c_1)$ (Putting $c = c_1$ in the rewritten series and that of its derivative with respect to c gives two independent solutions).

Summary

1. Assume series of the form:
$$y = x^c \left(a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r + \ldots\right)$$
2. Indicial equation gives $c = c_1$ and $c = c_2$.
3. **Case 1:** c_1 and c_2 differ by a quantity **NOT** *an integer*.
$\Longrightarrow$ Substitute $c = c_1$ and $c = c_2$ in the series for y.
4. **Case 2:** c_1 and c_2 deffer by *an* ***integer***, and a coefficient is *indeterminate* when $c = c_1$.
$\Longrightarrow$ Substitution of $c = c_1$ gives the complete solution.
5. **Case 3:** c_1 and c_2 $(c_1 < c_2)$ differ by *an* ***integer***, and a coefficient is *infinite* for $c = c_1$.
$\Longrightarrow$ Replace a_0 by $k(c - c_1)$. Then, put $c = c_1$ in the new series for y and for $\frac{\partial y}{\partial c}$.
$\Longrightarrow$ In general, if $c_1 - c_2 = n$ where $n \in \mathbb{Z} \neq 0$, the solution is of the form:
$$y = (1 + k \ln x)\, x^{c_1} \left\{a_0 + a_1 x + a_2 x^2 + \ldots\right\} + x^{c_2} \left\{b_0 + b_1 x + b_2 x^2 + \ldots\right\}$$
6. **Case 4:** c_1 and c_2 are *equal*.
$\Longrightarrow$ Substitute $c = c_1$ in the series for y and for $\frac{\partial y}{\partial c}$. Then make the substitution after differentiating.
$\Longrightarrow$ In general, if $c_1 = c_2 = c$, the solution is of the form:
$$y = (1 + k \ln x)\, x^c \left\{a_0 + a_1 x + a_2 x^2 + \ldots\right\} + x^c \left\{b_1 x + b_2 x^2 + \ldots\right\}$$

4.3 Bessel's Equation

Is a 2^{nd} order ODE that occurs frequently in STEM. Defined as :

$$\boxed{x^2y'' + xy' + \left(x^2 - v^2\right)y = 0} \quad \text{where } v \in \mathbb{R} \tag{4.12}$$

Starting with $y = x^c\left\{a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_rx^r + \ldots\right\}$ and proceeding as before, we obtain value of constants:

$$c = \pm v \quad \text{and} \quad a_1 = 0 \tag{4.13}$$

The recurrence relation is:

$$a_r = \frac{a_{r-2}}{v^2 - (c+r)^2}, \quad \text{for } r \geq 2 \tag{4.14}$$

Table 4.2: Values of Coefficients of Bessel's Function

Num.	a_n	Equation	When $c = +v$
1.	a_1	0	0
2.	a_2	$\dfrac{a_0}{v^2-(c+2)^2}$	$\dfrac{-a_0}{2^2 \times 1!\,(v+1)}$
3.	a_3	0	0
4.	a_4	$\dfrac{a_0}{\left[v^2-(c+2)^2\right]\left[v^2-(c+4)^2\right]}$	$\dfrac{a_0}{2^4 \times 2!\,(v+1)(v+2)}$
5.	a_5	0	0
6.	a_6	$\dfrac{a_0}{\left[v^2-(c+2)^2\right]\left[v^2-(c+4)^2\right]\left[v^2-(c+6)^2\right]}$	$\dfrac{-a_0}{2^6 \times 3!\,(v+1)(v+2)(v+3)}$
7.	a_7	0	0
8.	a_r	In general, when r =even	$\dfrac{(-1)^{r/2}a_0}{2^r \times \left(\frac{r}{2}\right)!\,(v+1)(v+2)\ldots\left(v+\frac{r}{2}\right)}$

When $\mathbf{c} = +\mathbf{v}$, the resulting series solution:

$$y = u = Ax^v\left[1 - \frac{x^2}{2^2 \times 1!\,(v+1)} + \frac{x^4}{2^4 \times 2!\,(v+1)(v+2)} - \frac{x^6}{2^6 \times 3!\,(v+1)(v+2)(v+3)} + \ldots\right] \tag{4.15}$$

When $\mathbf{c} = -\mathbf{v}$, the resulting series solution:

$$y = w = Bx^{-v}\left[1 - \frac{x^2}{2^2 \times 1!\,(v-1)} + \frac{x^4}{2^4 \times 2!\,(v-1)(v-2)} - \frac{x^6}{2^6 \times 3!\,(v-1)(v-2)(v-3)} + \ldots\right] \tag{4.16}$$

The complete solution (arbitrary constants A and B) to the Bessel's equation is :

$$y = u + w \tag{4.17}$$

4.3.1 Bessel Functions

It is convenient to present the solution in terms of **Gamma Functions** $\Gamma(x)$, $(x > 0)$. Assign a_0 the arbitrary value:

$$a_0 = \frac{1}{2^v \Gamma(v+1)} \tag{4.18}$$

We can rewrite terms in the Bessel Equation in therms of $\Gamma(x)$. Consider a_2 and $c = v$:

$$\begin{aligned} a_2 &= \frac{a_0}{v^2 - (c+2)^2} = \frac{a_0}{(v-c-2)(v+c+2)} = \frac{a_0}{-2(2v+2)} \\ &= \frac{-1}{2^2(v+1)} \cdot \frac{1}{2^v \Gamma(v+1)} = \frac{-1}{2^{v+2}(1!)\Gamma(v+2)} \end{aligned}$$

The recurrence relation is:

$$a_r = \frac{(-1)^{\frac{r}{2}}}{2^{v+r}\left(\frac{r}{2}!\right)\Gamma\left(v+\frac{r}{2}+1\right)} \xrightarrow{\text{Let } r=2k} \therefore a_{2k} = \frac{(-1)^k}{2^{v+2k}(k!)\Gamma(v+k+1)} \tag{4.19}$$

$$\text{where } r = \text{even and } k = 1, 2, 3, \ldots$$

Therefore, we can write the new form of series y:

$$y = x^v \left\{ \frac{1}{2^v \Gamma(v+1)} - \frac{x^2}{2^{v+2}(1!)\Gamma(v+2)} + \frac{x^4}{2^{v+4}(2!)\Gamma(v+3)} + \ldots \right\} \tag{4.20}$$

i) Bessel Function, Not Integers (v and $-v \notin \mathbb{Z}$)

$J_v(x)$ and $J_{-v}(x)$ are two independent solutions provided $v \notin \mathbb{Z}^-$ and $v \notin \mathbb{Z}^+$ respectively.

1. In general, Bessel function of the first kind order v, $J_v(x)$:

$$\left.\begin{aligned} J_v(x) &= \left(\frac{x}{2}\right)^v \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(v+k+1)} \\ J_{-v}(x) &= \left(\frac{x}{2}\right)^{-v} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(k-v+1)} \end{aligned}\right\} \text{ where } k = 0, 1, 2, 3, \ldots \tag{4.21}$$

2. The complete solution is a linear combination of $J_v(x)$ and $J_{-v}(x)$:

$$y = AJ_v(x) + BJ_{-v}(x) \quad \text{where } A, B = \text{constants}$$

3. Convergence of series for $\forall x$ can be established by normal ratio test.

ii) Bessel Function, Positive Integer ($v \in \mathbb{Z}^+$)

When $v = n$ (integer), then $J_v(x) \to J_n(x)$ and $J_{-v}(x) \to J_{-n}(x)$ are **NOT** *independent solutions*. Denoting positive $+v$ by n, we find that:

1. Using recurrence relation of Gamma Functions, $\Gamma(n+k+1) = (n+k)!$:
$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\,\Gamma(n+k+1)} = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\,(n+k)!} \tag{4.22}$$

2. The two solutions $J_n(x)$ and $J_{-n}(x)$ are related by:
$$J_{-n}(x) = (-1)^n J_n(x)$$

3. Therefore, applying the above, the final series for $J_n(x)$ is just:
$$J_n(x) = \left(\frac{x}{2}\right)^n \left\{\frac{1}{n!} - \frac{1}{(1!)(n+1)!}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(n+2)!}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)(n+3)!}\left(\frac{x}{2}\right)^6 + \ldots\right\} \tag{4.23}$$

4. The two commonly used functions:
$$J_0(x) = \left\{1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \ldots\right\}$$
$$J_1(x) = \frac{x}{2}\left\{1 - \frac{1}{(1!)(2!)}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(3!)}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)(4!)}\left(\frac{x}{2}\right)^6 \ldots\right\}$$

5. The complete solution is just a constant multiplied by $J_n(x)$:
$$y = CJ_n(x) \quad \text{where } C = \text{constant} \tag{4.24}$$

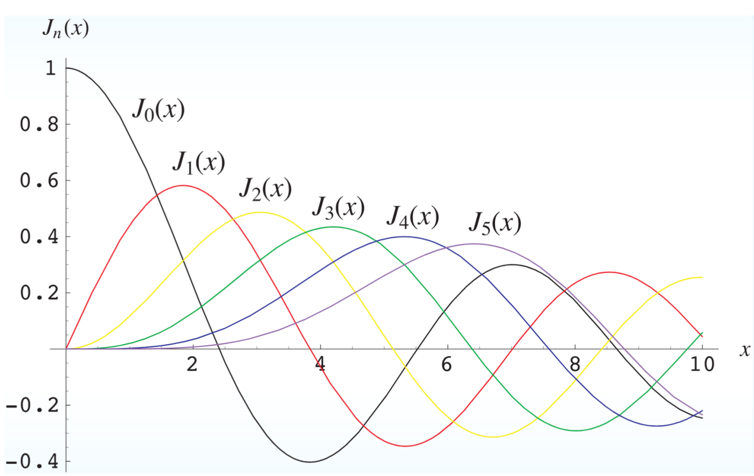


Figure 4.1: Graph of Bessel Function $J_0(x)$ and $J_1(x)$

4.4 Legendre's Equation

Another 2nd order ODE that occurs frequently in STEM. Defined as:

$$\left(1-x^2\right)y''-2xy'+k\left(k+1\right)y=0,\quad \text{where } k\in\mathbb{R} \tag{4.25}$$

Solving using Frobenius method, the indicial equation gives $c=0$ and $c=1$. The two corresponding solutions are:

$$[c=0]:\quad y_1=a_0\left\{1-\frac{k\left(k+1\right)}{2!}x^2+\frac{k\left(k-2\right)\left(k+1\right)\left(k+3\right)}{4!}x^4-\dots\right\} \tag{4.26}$$

$$[c=1]:\quad y_2=a_1\left\{x-\frac{\left(k-1\right)\left(k+2\right)}{3!}x^3+\frac{\left(k-1\right)\left(k-3\right)\left(k+2\right)\left(k+4\right)}{5!}x^5-\dots\right\} \tag{4.27}$$

where a_0 and a_1 are arbitrary constants

The *general solution* is thus $y=y_1+y_2$.

4.4.1 Legendre Polynomials

- When $k\in\mathbb{Z}$ (integer), one of the solution's series will terminate after finite terms.
- The polynomial that is left, $P_n\left(x\right)$ is called ***Legendre Polynomial.***
- Constants a_0 and a_1 are chosen such that $P_n\left(x\right)$ has unit value, $\left|P_n\left(x\right)\right|=1$ at $x=1$.
- For example, to find $P_2\left(x\right)$:

$$\begin{aligned}[c=0]:\quad y&=a_0\left\{1-\frac{2\left(3\right)}{2!}x^2+\frac{2\left(2-2\right)\left(3\right)\left(6\right)}{4!}x^4+\dots\right\}\\&=a_0\left\{1-3x^2\right\}\end{aligned}$$

When $a_0=-\frac{1}{2}$ and $x=1$, the value of $y=1$. Therefore:

$$P_2\left(x\right)=-\frac{1}{2}\left(1-3x^2-1\right)$$

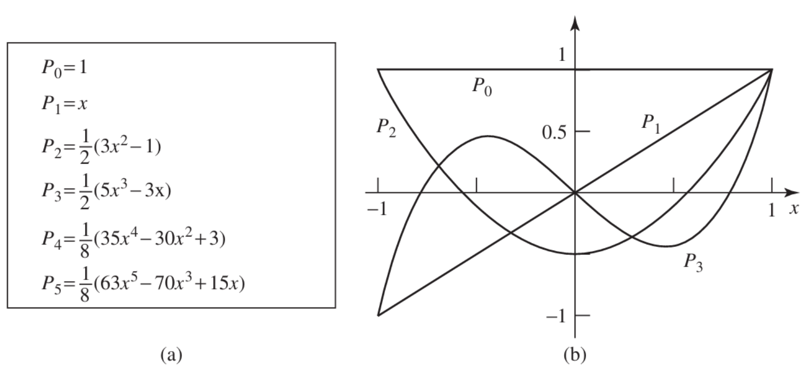


Figure 4.2: First few Legendre Polynomials, $P_n\left(x\right)$: (a) functional form, (b) graph

4.4.2 Rodrigue's Formula and the Generating Function

Legendre Polynomials can be derived by using ***Rodrigue's Formula:***

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n \tag{4.28}$$

Generating Function for Legendre Polynomials are useful to obtain values of $P_n(x)$.

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n, \quad |x| \leq 1,\ |t| < 1 \tag{4.29}$$

1. $P_n(-1)$ is found by considering $g(-1,t)$. Setting $x=-1$, we have:

   $$g(-1,t) = \frac{1}{\sqrt{1+3t+t^2}} \equiv \sum_{n=0}^{\infty} P_n(-1)\, t^n = P_0(0) + P_1(x)\, t + P_2(x)\, t^2 + P_3(x)\, t^3 + \ldots$$

   Using binomial expansion, we can expand $\mathcal{LHS}$:

   $$\frac{1}{\sqrt{1+3t+t^2}} = \frac{1}{1+t} = 1 - t + t^2 - t^3 + \ldots$$

   Therefore, comparing these expansions:

   $$P_n(-1) = (-1)^n.$$

2. $P_n(0)$ is found by considering $g(0,t)$. Setting $x=0$, we have:

   $$g(0,t) = \frac{1}{\sqrt{1+t^2}} \equiv \sum_{n=0}^{\infty} P_n(0)\, t^n = P_0(0) + P_1(0)\, t + P_2(0)\, t^2 + +P_3(0)\, t^3 \ldots$$

   Using binomial expansion, we can expand the $\mathcal{LHS}$:

   $$\frac{1}{\sqrt{1+t^2}} = 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \frac{5}{16}t^6 + \ldots$$

   Therefore, comparing these expansions:

   $$P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}$$

   where $n!!$ is the double factorial:

   $$n!! = \begin{cases} n(n-2)\ldots(3)\,1, & n>0,\ \text{odd}, \\ n(n-2)\ldots(4)\,2, & n>0,\ \text{even}, \\ 1, & n=0,\ -1 \end{cases}$$

4.4.3 Sturm-Liouville Systems

Sturm-Liouville system is a boundary value problem that is described by DE of form:

$$[p(x)\,y']' + [q(x) + \lambda r(x)]\,y = 0, \quad \text{for } a \le x \le b \text{ and } r(x) > 0 \tag{4.30}$$

where the boundary conditions can be written in the form

$$a_1 y(a) + a_2 y'(a) = 0 \quad \text{and} \quad \beta_1 y(b) + \beta_2 y'(b) = 0 \tag{4.31}$$

Solutions of such systems are in the form of an *infinite sequence of* ***eigenfunctions*** $\mathbf{y_n}$, each *corresponding to* an ***eigenvalue*** $\lambda_\mathbf{n}$, for $n = 0, 1, 2, \ldots$

For example, consider DE:

$$y'' + \lambda y = 0, \quad \text{for } 0 \le x \le 5 \xrightarrow[\text{conditions}]{\text{boundary}} y(0) = 0,\ y(5) = 0$$

The boundary equations imply:

$$\begin{aligned} \alpha_1 \cdot (0) + \alpha_2 y'(0) = 0 \quad &\text{and} \quad \beta_1 \cdot (0) + \beta_2 y'(0) = 0 \\ \therefore \alpha_2 = 0 \quad &\text{and} \quad \therefore \beta_2 = 0 \end{aligned}$$

Expand Equation 4.30 and comparing it with given DE:

$$\begin{gathered} p(x)\,y'' + p'(x)\,y' + [q(x) + \lambda r(x)] \equiv y'' + \lambda y = 0 \\ \therefore p'(x) = 0 \quad \text{and} \quad \therefore q(x) = 0 \quad \text{and} \quad \therefore r(x) = 1 \end{gathered}$$

To solve $y'' + \lambda y = 0$, use auxiliary equation $m^2 + \lambda = 0$, which gives us roots $m = i \pm \sqrt{\lambda}$. Therefore, the general solution is:

$$y = A \sin \sqrt{\lambda} x + B \cos \sqrt{\lambda} x$$

Applying boundary conditions $y(0) = 0$:

$$y(0) = 0 = A \sin(0) + B \cos(0) \Longrightarrow B = 0$$

Applying boundary conditions $y(5) = 0$:

$$y(5) = 0 = A \sin\left(5\sqrt{\lambda}\right) \Longrightarrow \sqrt{\lambda} = \frac{n\pi}{5} \to \lambda = \frac{n^2\pi^2}{25}$$

There is ∞ number of eigenvalues, λ. The n^{th} eigenvalue being denoted by λ_n where $\lambda_n = \dfrac{n^2\pi^2}{25}$, with each eigenvalue having its corresponding eigenvector solution, $y_n = A_n \sin \dfrac{n\pi x}{5}$.

4.4.4 Orthogonality

Two functions, $f(x)$ and $g(x)$ defined on interval $a \leq x \leq b$ are **mutually orthogonal** if:

$$\int_a^b f(x)\, g(x)\, dx = 0 \tag{4.32}$$

Meanwhile, two functions are **mutually orthogonal** with **respect to the weight function** $w(x)$ if there's a *third function* $w(x) > 0$ *exists* such that:

$$\int_a^b f(x)\, g(x)\, w(x)\, dx = 0 \tag{4.33}$$

An **important property** of the **solutions** to **Sturm-Liouville system** is that *all solutions are mutually orthogonal with respect to weight function* $r(x)$.

$$\int_a^b y_m(x)\, y_n(x)\, r(x)\, dx = 0, \quad (m \neq n) \tag{4.34}$$

4.4.5 Revisited Legendre's Equation

All **Legendre Polynomials,** $P_n(x)$ are *mutually orthogonal*:

$$\int_{-1}^1 P_m(x)\, P_n(x)\, dx = 0 \tag{4.35}$$

Proof:
The equations $(1-x^2)\, y'' - 2xy' + n(n+1)\, y = 0$ is Legendre's Equation and has Legendre Polynomials, $P_n(x)$ as solutions:

$$y_n = P_n(x), \quad \text{where } P_n(1) = 1 \text{ and } P_n(-1) = (-1)^n$$

This equation is an example of Sturm-Liouville system $[p(x)\, y']' + [q(x) + \lambda r(x)]\, y = 0$ with boundary conditions $a_1 y(a) + a_2 y'(a) = 0$ and $\beta_1 y(b) + \beta_2 y'(b) = 0$ where:

$$p(x) = 1 - x^2 \quad \text{and} \quad q(x) = 0 \quad \text{and} \quad r(x) = 1 \quad \text{and} \quad \alpha_1, \alpha_2 = 1, 0 \quad \text{and} \quad \beta_1, \beta_2 = 1, 0$$

Consequently, Legendre Polynomials $P_n(x)$ are mutually orthogonal when $m \neq n$.

4.4.6 Polynomials as Finite Series of Legendre Polynomials

- Many DE cannot be solved analytically, so solution by power series is a powerful tool.
- **Any polynomial** can be *written* in a *finite series of Legendre Polynomials,* $P_n(x)$.
- Example 1: Show that $f(x) = x^2$ can be written as a series of Legendre Polynomials. Assume that,

$$\begin{aligned} f(x) = x^2 &= \sum_{n=0}^{\infty} a_n P_n(x), \text{ then} \\ x^2 &= a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \dots \\ &= a_0(1) + a_1(x) + a_2 \frac{3x^2 - 1}{2} + a_3 \frac{5x^3 - 3x}{2} + \dots \end{aligned}$$

Since $\mathcal{LHS}$ is a polynomial of degree $= 2$, any Legendre Polynomial on $\mathcal{RHS}$ containing power of $x > 2$ is excluded, i.e. $a_3 = a_4 = \ldots = 0$. Therefore:

$$x^2 = a_0 - \frac{a_2}{2} + a_1 x + \frac{3}{2} a_2 x^2 \implies a_2 = \frac{2}{3},\ a_1 = 0,\ a_0 - \frac{a_2}{2} = 0 \to a_0 = \frac{1}{3}$$

Finally, we obtain an expression in terms of Legendre Polynomial, $P_n\left(x\right)$:

$$x^2 = \frac{1}{3} P_0\left(x\right) + \frac{2}{3} P_2\left(x\right)$$